SiRF White Paper

Great Circle Distances

Computing the distance between two points on the surface of the Earth

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Introduction

The distance between two points on the Earth is computed using a Great Circle, that is, the shortest distance which on a sphere is along the path of a circle whose center is the center of the sphere. Since the Earth is modeled not as a sphere but as an ellipsoid, computing this distance can get quite complicated. In this document several methods are presented, from the simple (and intuitive) to the more precise. But based on tests done to compare all the methods, the actual difference between the methods is less than 1% over distances less than a few hundred kilometers. These calculations do not incorporate altitude.

Common Definitions

In this paper, the following symbols have a common meaning:

φ	Latitude
λ	Longitude
(φ,λ)	Lat/Lon for a point, subscripts are used to distinguish between points
R_E	Radius of the Earth (use 6,378,137 m, from WGS-84 semi-major axis, a)
f	Flattening of the Earth (use 1/298.257223563, from WGS-84)
b	Semi-minor axis of the Earth ellipsoid, $b = a(1 - f)$, where $a = R_E$
$\Delta\varphi$, $\Delta\lambda$	Change or difference in latitude, longitude
Northing	A distance north (or south, for negative values) from a reference point
Easting	A distance east (or west for negative values) from a reference point
$\Psi_{a,b}$	Bearing from point a to point b
GCD	Great Circle Distance

Method 1: Intuitive

The first method works based on two assumptions: that the Earth is a perfect sphere, and that in the region of the two points it is basically a plane. These assumptions introduce errors that are greater as the distance between the points increases, but over a couple hundred kilometers it is still well below 1 %.

Assume that the Earth is a perfect sphere with radius R_E. With this assumption, a great circle on the sphere has circumference = $2\pi R_E$ = 40 075 016.69 m. Likewise, a circle of latitude has circumference = $2\pi R_E \cos(\varphi)$.

Next compute Northing and Easting between the two points. Select one point as the reference, with coordinates (φ_1, λ_1). The second point, (φ_2, λ_2) is displaced from the first point by $\Delta\varphi$, $\Delta\lambda$, where $\Delta\varphi = \varphi_2 - \varphi_1$ and $\Delta\lambda = \lambda_2 - \lambda_1$. Now,

Northing	$= \Delta\varphi / 360 * 2\pi R_E$	($\Delta\varphi$ in degrees)
Easting	$= \Delta\lambda / 360 * 2\pi R_E \cos(\varphi_1)$	($\Delta\lambda$ in degrees)

These equations simply relate the change in latitude or longitude to fractions of a circle ($\Delta\varphi / 360$, where $\Delta\varphi$ is in degrees, or $\Delta\varphi / 2\pi$, where $\Delta\varphi$ is in radians) times the circumference of that circle.

Next, assume that locally the two points are on a plane, and that the Northing and Easting represent two legs of a right triangle. The hypotenuse of that triangle is the distance between the two points. Thus:

$$\text{GCD} = (\text{Northing}^2 + \text{Easting}^2)^{1/2} \quad \text{units are the same as } R_E$$

Method 1A: Intuitive with Accomodations of the Ellipsoid

Modify Method 1 by using the semi-minor axis of the Earth reference ellipsoid as the radius of the great circle of longitude. For the circle of latitude, continue to use the semi-major axis and the cosine of the latitude. This implies that the circle of longitude is a circle rather than an ellipse, but the difference is smaller than the difference between using a circle with radius a and the ellipse.

$b = a\,(1 - f)$		definition of semi-minor axis
$b = 6378137 * (1 - 1 / 298.257223563)$		use WGS-84 values
$b = 6356752.314$		in meters
Northing	$= \Delta\varphi / 360 * 2\pi b$	($\Delta\varphi$ in degrees)
Easting	$= \Delta\lambda / 360 * 2\pi R_E \cos(\varphi_1)$	($\Delta\lambda$ in degrees)
GCD	$= (\text{Northing}^2 + \text{Easting}^2)^{1/2}$	in meters

Method 2: Geometric, Assuming a Spherical Earth

This method comes from books on aircraft guidance. It is a bit more accurate, but still assumes that the Earth is a perfect sphere of radius R_E. The error in this assumption can be quantified by comparing the radius assumed with the actual length of a great circle of longitude, where the radius is the semiminor axis of the Earth, b. b can be computed using the WGS-84 definition where b = a (1 – f). f is defined as 1/298.257223563, so b is seen to be approximately 6354752.314 m, thus differing from the assumed R_E by 23384.686 m, or 0.367 %.

Great Circle Distance from point 2 to point 1, where (φ_2, λ_2) and (φ_1, λ_1) represent points 2 and 1, respectively.

$$\Delta\varphi = \varphi_2 - \varphi_1$$
$$\Delta\lambda = \lambda_2 - \lambda_1$$
$$X = \cos(\varphi_2)\sin(\Delta\lambda)$$
$$Y = \sin(\Delta\varphi) + \cos(\varphi_2)\sin(\varphi_1)[1 - \cos(\Delta\lambda)]$$
$$\Psi_{2,1} = \tan^{-1}(X/Y) \quad \text{(= Bearing from point 2 to point 1)}$$
$$P = Y\cos(\Psi_{2,1}) + X\sin(\Psi_{2,1})$$
$$1 + Z = \cos(\Delta\varphi) - \cos(\varphi_1)\cos(\varphi_2)[1 - \cos(\Delta\varphi)]$$
$$\Theta_{2,1} = \tan^{-1}[P / (1 + Z)]$$
$$GCD = R_E\,\Theta_{2,1}$$

Method 3: Geometric Assuming an Ellipsoidal Earth

This method comes from a Magnavox reference on navigation and should be the most precise of all presented here.

$\Delta\lambda = \lambda_2 - \lambda_1$

$t_1 = \cos\varphi_2 \cos\Delta\lambda$

$K = \sin\varphi_2 \cos\varphi_1 - t_1 \sin\varphi_1$

$t_2 = \cos\varphi_2 \sin\Delta\lambda$

$\cos c = \sin\varphi_1 \sin\varphi_2 + t_1 \cos\varphi_1$

$\sin c = (t_2^2 + K^2)^{1/2}$

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if (sin c == 0)
        Ψ1,2 = 0
else
        L = sin^-1 (t2 / sin c)            in radians
        if (K >= 0)
                M = L
        else if (K < 0)
                M = π – L
        if (M >= 0)
                Ψ1,2 = M
        else
                Ψ1,2 = M + 2 π
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$c = \sin^{-1}(\sin c)$

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if (cos c > 0)
        d = c
else
        cos c = - cos c
        d = π – c
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$$GCD = R_E\{ d - f/4 [(d + 3 \sin c)/(2 \sin^2 (d/2))*(\sin\varphi_1 - \sin\varphi_2)^2 + (d - 3 \sin c)/(1 + \cos c)* (\sin\varphi_1 + \sin\varphi_2)^2]\}$$